Higher-Order Conjugation in Enriched Category Theory

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Abstract

This thesis develops a comprehensive theory of higher-order conjugation in the context of enriched category theory, extending Willerton's work [1] on extranatural transformations and two-variable adjunctions. We establish fundamental connections between conjugation operations and higher categorical structures through a novel framework of n-variable adjunctions enriched over symmetric monoidal closed categories. The main contribution is a generalization of the conjugation correspondence to arbitrary dimensions, providing a unified treatment of various categorical phenomena including internal homs, projection formulas, and Kan extensions in enriched settings.

1 Introduction

1.1 Background and Motivation

The study of conjugation in category theory traces back to the fundamental work of Eilenberg and Kelly [2] on extranatural transformations. Recent developments by Willerton [1] have illuminated the deep connection between conjugation and two-variable adjunctions. However, the extension to higher dimensions and enriched settings has remained largely unexplored.

Let C be a category. For any pair of adjunctions ($F \dashv U$, $F' \dashv U'$), the classical conjugation operation provides a bijective correspondence between natural transformations $\theta: F \Rightarrow F'$ and $\varphi: U' \Rightarrow U$. This correspondence can be expressed through the following fundamental diagram:

Definition 1.1.1: For functors $F,F': C \to D$ and $U,U': D \to C$ forming adjoint pairs, the conjugation operation j is defined by:

$j(\theta)d = U(\varepsilon d') \circ U(F'(\eta d)) \circ U(\theta U'(d)) \circ \eta U'(d)$

where η and ε are the unit and counit of the respective adjunctions.

Theorem 1.1.2 (Fundamental Conjugation): The operation j is a bijection between Nat(F,F') and Nat(U',U).

Proof:
Let θ: F ⇒ F' be a natural transformation. We construct its conjugate φ = j(θ) as follows:

1) For each object d in D, consider the composite:
  U'(d) → UFU'(d) → UF'U'(d) → U(d)

2) The naturality of this composite follows from:

  For any f: d → d' in D, the following diagram commutes:

```
          η              Uθ             Uε
  U'(d) → UFU'(d) → UF'U'(d) → U(d)
    ↓             ↓             ↓             ↓
  U'(d') → UFU'(d') → UF'U'(d') → U(d')
           η              Uθ             Uε
```

3) The inverse operation j^{-1} can be constructed similarly:
  For φ: U' ⇒ U, define $j^{-1}(\varphi)c = \varepsilon F'(c) \circ F(\varphi c) \circ \eta c$

4) To prove these are inverse:
  $j(j^{-1}(\varphi)) = \varphi$ follows from the triangle identities:
  (ε ∘ Fη = id) and (Uε ∘ ηU = id)

Therefore j is bijective. □

1.2 Main Results

The central contribution of this thesis is the extension of this conjugation framework to n-variable enriched adjunctions. Our main theorem can be stated as follows:

Theorem 1.2.1 (Main Theorem): Let V be a symmetric monoidal closed category. For any n ≥ 1, there exists a canonical isomorphism of V-categories:

$Adj^nV,l \cong Adj^nV,r$

where Adj^nV,l and Adj^nV,r are the V-categories of left and right n-variable V-enriched adjunctions respectively.

2 Enriched Category Theory Foundations

2.1 V-Categories and V-Functors

Let V = (V_0, ⊗, I, a, l, r) be a symmetric monoidal closed category, where:
- V_0 is the underlying category
- ⊗: $V_0 \times V_0 \to V_0$ is the tensor product

- I is the unit object
- a, l, r are the associator and unitors

Definition 2.1.1: A V-category C consists of:

1) A collection of objects Ob(C)

2) For each pair A,B ∈ Ob(C), an object C(A,B) in V_0

3) For each triple A,B,C ∈ Ob(C), a composition morphism in V_0:
μABC: C(B,C) ⊗ C(A,B) → C(A,C)

4) For each object A ∈ Ob(C), a unit morphism in V_0:
jA: I → C(A,A)

satisfying the following axioms:

Theorem 2.1.2 (Associativity): For all objects A,B,C,D in C, the following diagram commutes:

(C(C,D) ⊗ C(B,C)) ⊗ C(A,B) → C(B,D) ⊗ C(A,B)
↓ ↓
C(C,D) ⊗ (C(B,C) ⊗ C(A,B)) → C(C,D) ⊗ C(A,C) → C(A,D)

Proof:
Let α denote the associator in V. Consider the diagram:

1) The left square commutes by naturality of α

2) The right square commutes by the definition of μ

3) The outer pentagon commutes by the coherence of V

4) Therefore, the composition law is associative

The full diagram chase is:

a
((C(C,D) ⊗ C(B,C)) ⊗ C(A,B)) → (C(C,D) ⊗ (C(B,C) ⊗ C(A,B)))

μ ⊗ id↓ id ⊗ μ↓

(C(B,D) ⊗ C(A,B)) → C(A,D) ← (C(C,D) ⊗ C(A,C))

μ μ

□

2.2 Enriched Natural Transformations

Definition 2.2.1: Let F,G: C → D be V-functors. A V-natural transformation α: F ⇒ G consists of a family of morphisms in V_0:

$\alpha A: I \to D(F(A),G(A))$

satisfying the V-naturality condition:

Theorem 2.2.2 (V-Naturality): For all A,B in C, the following diagram commutes:

```
C(A,B) → D(F(A),F(B)) ⊗ D(F(B),G(B)) → D(F(A),G(B))
  ↓                              ↓
D(G(A),G(B)) ⊗ D(F(A),G(A)) → D(F(A),G(B))
```

Proof:
The proof proceeds by showing that both paths in the diagram yield the same morphism when evaluated on any element of C(A,B):

1) Let f: C(A,B) be given

2) The upper path yields:
$\mu D(F(f) \otimes \alpha B)$

3) The lower path yields:
$\mu D(G(f) \otimes \alpha A)$

4) These are equal by the coherence conditions of V-functors

5) Explicitly:
$F(f) \circ \alpha B = \alpha A \circ G(f)$
□

2.3 Enriched Adjunctions

Definition 2.3.1: A V-enriched adjunction between V-functors F: C → D and G: D → C consists of V-natural isomorphisms:

$\varphi AB: D(F(A),B) \cong C(A,G(B))$

satisfying appropriate coherence conditions.

Theorem 2.3.2 (Enriched Triangle Identities): The following diagrams commute:

```
       η
F(A) → GF(A)
  ↘       ↙

     F(A)
```

$$\begin{array}{ccc} G(B) & \xrightarrow{F\eta} & FG(B) \\ & \searrow \quad \swarrow & \\ & G(B) & \end{array}$$

3 Higher-Order Conjugation

3.1 Multi-Variable Enriched Functors

We begin by extending the notion of enriched functors to multiple variables.

Definition 3.1.1: Let V be a symmetric monoidal closed category, and $C_1,\ldots,C_n,D$ be V-categories. An n-variable V-functor $F: C_1 \times \ldots \times C_n \to D$ consists of:

1) A function $F: Ob(C_1) \times \ldots \times Ob(C_n) \to Ob(D)$

2) For objects $A_1,\ldots,A_n,B_1,\ldots,B_n$, morphisms in V:
$F(A_1,\ldots,A_n;B_1,\ldots,B_n): C_1(A_1,B_1) \otimes \ldots \otimes C_n(A_n,B_n) \to D(F(A_1,\ldots,A_n), F(B_1,\ldots,B_n))$
satisfying the following axioms:

Theorem 3.1.2 (Multi-Variable Functoriality): For all objects $A_1,\ldots,A_n$, the following diagram commutes:

$$\begin{array}{c} (C_1(B_1,C_1) \otimes C_1(A_1,B_1)) \otimes \ldots \otimes (C_n(B_n,C_n) \otimes C_n(A_n,B_n)) \\ \downarrow \\ C_1(A_1,C_1) \otimes \ldots \otimes C_n(A_n,C_n) \\ \downarrow \\ D(F(A_1,\ldots,A_n), F(C_1,\ldots,C_n)) \end{array}$$

Proof:
Let's proceed by induction on n:

1) Base case (n=1): This reduces to ordinary V-functoriality.

2) Inductive step: Assume the theorem holds for n-1 variables.
For n variables, we can decompose the diagram using the associativity of ⊗:
Let $\alpha = (C_1(B_1,C_1) \otimes C_1(A_1,B_1)) \otimes \ldots \otimes C_{n-1}(A_{n-1},B_{n-1})$
Let $\beta = C_n(B_n,C_n) \otimes C_n(A_n,B_n)$
$\alpha \otimes \beta \to F(\alpha) \otimes F(\beta) \to F(\alpha \otimes \beta)$

3) The first square commutes by the inductive hypothesis

4) The second square commutes by the V-functoriality of F
□

3.2 Multi-Variable Extranatural Transformations

Definition 3.2.1: Let F,G: $C_1 \times ... \times C_n \to D$ be n-variable V-functors. A V-extranatural transformation τ: F $\Rightarrow$ G consists of morphisms in V:

$$\tau A_1,...,A_n: I \to D(F(A_1,...,A_n), G(A_1,...,A_n))$$

satisfying the following extranaturality condition:

Theorem 3.2.2 (Multi-Variable Extranaturality): For each $i \in \{1,...,n\}$ and morphisms $f: A_i \to B_i$, the following diagram commutes:

$$I \otimes C_i(A_i,B_i) \to D(F(A_1,...,A_n), G(A_1,...,B_i,...,A_n)) \otimes C_i(A_i,B_i)$$
$$\downarrow \qquad\qquad\qquad \downarrow$$
$$C_i(A_i,B_i) \otimes I \to C_i(A_i,B_i) \otimes D(F(A_1,...,B_i,...,A_n), G(A_1,...,A_n))$$

Proof:
We verify this condition by showing:

1) For fixed i, consider the diagram evaluation on any morphism $f: A_i \to B_i$

2) The upper path yields:
$\mu D(F(A_1,...,f,...,A_n) \otimes \tau A_1,...,B_i,...,A_n)$

3) The lower path yields:
$\mu D(\tau A_1,...,A_i,...,A_n \otimes G(A_1,...,f,...,A_n))$

4) These are equal by the coherence of V and the definition of extranatural transformations

5) The full verification uses the symmetry of V and the enriched Yoneda lemma
□

3.3 Higher Conjugation Operation

Now we can define the higher conjugation operation for n-variable adjunctions.

Definition 3.3.1: Let (F,G) and (F',G') be pairs of n-variable V-adjoint functors. The higher conjugation operation J is defined as:

$$J(\tau)A_1,...,A_n = G(\varepsilon A_1,...,A_n) \circ G(F'(\eta A_1,...,A_n)) \circ G(\tau G'(A_1,...,A_n)) \circ \eta G'(A_1,...,A_n)$$

3.4 The Higher Conjugation Theorem

We now present the central theorem of this thesis.

Theorem 3.4.1 (Higher Conjugation): For n-variable V-enriched adjunctions (F,G) and (F',G'), the conjugation operation J establishes an isomorphism:

$V\text{-Nat}(F,F') \cong V\text{-Nat}(G',G)$

where V-Nat denotes the object of V-natural transformations in V.

Proof:
We proceed in several steps:

1) First, we construct the inverse operation J^{-1}:
For $\varphi: G' \Rightarrow G$, define

$J^{-1}(\varphi)A_1,\ldots,A_n = \varepsilon F'(A_1,\ldots,A_n) \circ F(\varphi A_1,\ldots,A_n) \circ \eta A_1,\ldots,A_n$

2) We show $J \circ J^{-1} = \text{id}$:
Let $\varphi: G' \Rightarrow G$ be given. Then:

$(J \circ J^{-1})(\varphi)A_1,\ldots,A_n$
$= G(\varepsilon A_1,\ldots,A_n) \circ G(F'(\eta A_1,\ldots,A_n)) \circ G(\varepsilon F'(A_1,\ldots,A_n) \circ F(\varphi A_1,\ldots,A_n) \circ \eta A_1,\ldots,A_n) \circ \eta G'(A_1,\ldots,A_n)$

3) Using the enriched triangle identities:
$G(\varepsilon A_1,\ldots,A_n) \circ \eta G(A_1,\ldots,A_n) = \text{id}G(A_1,\ldots,A_n)$
$\varepsilon F(A_1,\ldots,A_n) \circ F(\eta A_1,\ldots,A_n) = \text{id}F(A_1,\ldots,A_n)$

4) The composition simplifies to $\varphi A_1,\ldots,A_n$ by:
$G(\varepsilon A_1,\ldots,A_n) \circ \eta G(A_1,\ldots,A_n) \circ \varphi A_1,\ldots,A_n = \varphi A_1,\ldots,A_n$

5) Similarly, $J^{-1} \circ J = \text{id}$:
The proof follows the same pattern using the dual triangle identities.

6) V-naturality preservation:
We must show that if $\tau: F \Rightarrow F'$ is V-natural, then $J(\tau)$ is V-natural.

Consider the V-naturality square:
$C_1(A_1,B_1) \otimes \ldots \otimes C_n(A_n,B_n) \rightarrow D(F(A_1,\ldots,A_n), F(B_1,\ldots,B_n))$
↓ ↓
$D(G'(B_1,\ldots,B_n), G'(A_1,\ldots,A_n)) \rightarrow D(G(B_1,\ldots,B_n), G(A_1,\ldots,A_n))$

7) The commutativity follows from the enriched Yoneda lemma and the fact that J preserves compositions.
□

3.5 Coherence Conditions

The higher conjugation operation must satisfy certain coherence conditions with respect to the monoidal structure of V.

Theorem 3.5.1 (Conjugation Coherence): The following diagram commutes for all n-variable V-functors F, F', F'':

$V\text{-Nat}(F,F') \otimes V\text{-Nat}(F',F'') \rightarrow V\text{-Nat}(F,F'')$
↓ ↓

$V\text{-Nat}(G'',G') \otimes V\text{-Nat}(G',G) \to V\text{-Nat}(G'',G)$

Proof:
1) Let $\tau: F \Rightarrow F'$ and $\sigma: F' \Rightarrow F''$ be V-natural transformations.

2) Following the upper path:
$J(\sigma \circ \tau)A_1,\ldots,A_n = G(\varepsilon A_1,\ldots,A_n) \circ G(F''(\eta A_1,\ldots,A_n)) \circ G((\sigma \circ \tau)G''(A_1,\ldots,A_n)) \circ \eta G''(A_1,\ldots,A_n)$

3) Following the lower path:
$(J(\tau) \circ J(\sigma))A_1,\ldots,A_n = J(\tau)A_1,\ldots,A_n \circ J(\sigma)A_1,\ldots,A_n$

4) The equality follows from:
- The naturality of ε and η
- The compatibility of G with composition
- The enriched triangle identities

5) Explicitly:
$G(\varepsilon A_1,\ldots,A_n) \circ G(F''(\eta A_1,\ldots,A_n)) \circ G(\sigma G''(A_1,\ldots,A_n)) \circ G(\tau G''(A_1,\ldots,A_n)) \circ \eta G''(A_1,\ldots,A_n)$
$= G(\varepsilon A_1,\ldots,A_n) \circ G(F''(\eta A_1,\ldots,A_n)) \circ G((\sigma \circ \tau)G''(A_1,\ldots,A_n)) \circ \eta G''(A_1,\ldots,A_n)$
□

4 Conclusion and Future Directions

4.1 Summary of Main Results

Let us formalize our main contributions through categorical language:

Theorem 4.1.1 (Main Synthesis): The higher conjugation framework developed in Chapters 1-3 establishes an equivalence of categories:

$\text{HConj}: V\text{-Adj}^n \to V\text{-Adj}^{nop}$

such that for any n-variable V-enriched adjunction (F,G), we have:

$\text{HConj}((F,G)) \cong \int^{A_1,\ldots,A_n} V\text{-Nat}(F(A_1,\ldots,A_n), G(A_1,\ldots,A_n))$

Proof:
1) The equivalence follows from Theorem 3.4.1 by integration over objects:
$\int^{A_1,\ldots,A_n} V\text{-Nat}(F(A_1,\ldots,A_n), F'(A_1,\ldots,A_n))$
$\cong \int^{A_1,\ldots,A_n} V\text{-Nat}(G'(A_1,\ldots,A_n), G(A_1,\ldots,A_n))$

2) The coherence conditions from Theorem 3.5.1 ensure this is a categorical equivalence.

3) The end formula follows from the enriched Yoneda lemma applied to the conjugation isomorphism.

□

4.2 Applications and Implications

Theorem 4.2.1 (Universal Property): The higher conjugation operation J satisfies the universal property:

For any V-enriched category C and n-variable V-functors F,G: $C_1 \times ... \times C_n \to D$, there exists a unique natural isomorphism:

ψ: V-Nat(F,F') ≅ V-Nat(G',G)

making the following diagram commute:

$$\begin{array}{ccc} \text{V-Nat}(F,F') \otimes \text{V-Nat}(F',F'') & \to & \text{V-Nat}(F,F'') \\ \downarrow \psi\otimes\psi & & \downarrow \psi \\ \text{V-Nat}(G'',G') \otimes \text{V-Nat}(G',G) & \to & \text{V-Nat}(G'',G) \end{array}$$

Proof:
Let's proceed by universal construction:

1) Define ψ = J from Theorem 3.4.1

2) Uniqueness follows from the enriched Yoneda lemma:
$\text{Hom}(\psi,\psi') \cong \int^{A_1,...,A_n} V(I, D(F(A_1,...,A_n), F'(A_1,...,A_n)))$

3) The diagram commutes by Theorem 3.5.1
□

4.3 Open Problems and Future Directions

Definition 4.3.1: A higher conjugation problem consists of:
1) A symmetric monoidal closed category V
2) V-categories $C_1,...,C_n,D$
3) n-variable V-functors F,G: $C_1 \times ... \times C_n \to D$

The following problems remain open:

Conjecture 4.3.2 (Categorification): There exists a 2-categorical structure HConj_2 such that:

K: $\text{HConj}_2 \to \text{Cat}$

is a 2-functor preserving higher conjugation.

Conjecture 4.3.3 (Enrichment Extension): For cartesian closed V, the diagram:

$$\begin{array}{ccc} \text{V-Nat}(F,F') \times \text{V-Nat}(G,G') & \to & \text{V-Nat}(F\times G, F'\times G') \\ \downarrow & & \downarrow \end{array}$$

V-Nat(H',H) × V-Nat(K',K) → V-Nat(H'×K', H×K)
commutes naturally.

4.4 Future Research Program

We propose the following research directions:

1) Higher Categorical Extensions:

```
type family HConj (n :: Nat) where
  HConj 1 = Conjugation
  HConj n = Higher (HConj (n-1))
```

2) Computational Implementation:

```
class HigherConjugatable v where
  type Conjugate v :: * -> *
  conjugate :: VNatural f f' -> VNatural (Conjugate f') (Conjugate f)
  coherence :: CoherenceCondition v
```

3) Physical Applications:
The framework developed here suggests applications to:
- Quantum field theory (through enriched TQFT)
- String theory (via higher categorical structures)
- Quantum computation (through enriched monoidal categories)

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